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On AdS/CFT of Galilean Conformal Field Theories

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Abstract

We study a new contraction of a $d+1$ dimensional relativistic conformal algebra where $n+1$ directions remain unchanged. For $n = 0, 1$ the resultant algebras admit infinite dimensional extension containing one and two copies of Virasoro algebra, respectively. For $n > 1$ the obtained algebra is finite dimensional containing an $so(2, n+1)$ subalgebra. The gravity dual is provided by taking a Newton-Cartan like limit of the Einstein's equations of AdS space which singles out an AdS_{n+2} spacetime. The infinite dimensional extension of $n = 0, 1$ cases may be understood from the fact that the dual gravities contain AdS_2 and AdS_3 factor, respectively. We also explore how the AdS/CFT correspondence works for this case where we see that the main role is playing by AdS_{n+2} base geometry.

1 Introduction

Non-relativistic AdS/CFT correspondence has recently been studied in several papers including [1] - [37]. Actually non-relativistic CFTs may be obtained from relativistic CFTs by making use of a *non-relativistic limit*. In general by taking a non-relativistic limit it means that we are sending the speed of light to infinity. More precisely we have $v/c \to 0$ where v is the typical speed of the model. We note, however, that there are several ways to take this limit which may even reduce the dimensions of the spacetime too.

To explore the procedures of taking the non-relativistic limit we will start from a relativistic CFT in $d+1$ dimensions parametrized by t, x_i for $i = 1, \cdots, d$. To proceed let us decompose the coordinates as $(x^+, x^-, x_i),\ i = 1, \cdots, d-1$, where the light like coordinates (x^+, x^-) are defined by

$$x^+ = \frac{1}{\sqrt{2}}(t + x_d), \qquad x^- = \frac{1}{\sqrt{2}}(t - x_d). \tag{1.1}$$

Next we compactify the light like coordinate x^- and identify the momentum along the light like coordinate with the number operator of the non-relativistic CFT. Then we look for those generators of the relativistic conformal algebra that commute with the number operator which altogether construct an algebra. The resultant algebra is the Schrödinger algebra [38, 39] which is the symmetry of the Schrödinger equation. In other words the Schrödinger group may be thought of as a subgroup of $SO(2, d+1)$ with fixed momentum along the null direction (see for example [40–45]). A theory with this symmetry is a non-relativistic CFT with the following scaling symmetry

$$x^+ \to \lambda^2 x^+, \qquad x_i \to \lambda x_i. \tag{1.2}$$

Note that starting from $d+1$ dimensional relativistic CFT the obtained theory is a non-relativistic CFT in d dimensions. This symmetry, for example, is relevant to study cold atoms [1]. The generators of the corresponding algebra are spatial translations P_i, rotations M_{ij}, time translation H, Galilean boosts B_i, dilation D, number operator N and special conformal transformation K. The algebra has also a central extension given by the number operator.

Recently gravity duals of non-relativistic CFTs have been proposed in [1, 2]. It has also been shown [46] that the asymptotic symmetry algebra of the corresponding geometry, in any dimension, is an infinite dimensional algebra containing one copy Virasoro algebra compatible with the symmetry of non-relativistic CFT [47].

On the other hand one may look for a non-relativistic conformal algebra which scales space and time in the same way

$$t \to \lambda t, \qquad x_i \to \lambda x_i \tag{1.3}$$

This algebra has recently been studied in [33](see also [48]) where it was shown that the corresponding algebra may be obtained from $d+1$ dimensional relativistic conformal algebra by making use of a contraction. Since the contraction does not change the dimension of the algebra the resultant algebra can be thought of as the symmetry of a non-relativistic CFT in $d+1$ dimensions. More precisely the contraction can be defined by the scaling

$t \to t$, $x_i \to \epsilon x_i$ in the limit of $\epsilon \to 0$. The generators of the obtained algebra are spatial translations, P_i, rotations J_{ij}, time translation H, Galilean boosts B_i, dilation D, special conformal transformation K and spatial special conformal transformation K_i.

It is also shown [33] that the corresponding algebra admits an infinite dimensional extension containing one copy of Virasoro algebra and the bulk gravity dual is provided by a Newtonian gravity given in terms of a non-dynamical metric but a dynamical torsion free affine connections. The gravity background may also be thought of as spatial $d-1$ dimensional space fibered over an AdS_2. In this sense the infinite dimensional extension may be understood from the asymptotic isometries of this AdS_2.

The aim of this article is to extend the above considerations for a new contraction in which we scale $d-n-1$ directions by ϵ and $n+1$ directions remain unchanged. Then we consider the limit of $\epsilon \to 0$. The obtained algebra which we call it *semi-Galilean algebra* can be thought of as a symmetry of non-relativistic CFT in $d+1$ dimensions[1].

We show that for $n=1$ the corresponding algebra admits an infinite dimensional extension containing two copies of Virasoro algebra. The corresponding gravity dual is defined on a geometry which is a $d-2$ dimensional spatial space fibered over an AdS_3 and, indeed, the infinite dimensional extension can be associated to the asymptotic isometries of AdS_3.

For $n \geq 2$ the contraction leads to an algebra which has $so(2,n+1) \times so(d-n-1)$ subalgebra and the corresponding gravity dual is defined by a geometry which is a $d-n-1$ dimensional spatial space fibered over an AdS_{n+2}. The subalgebra can, then, be identified with the isometries of AdS_{n+2}. Since the asymptotic symmetry algebra of AdS_{n+2} space for $n \geq 2$ is finite dimensional, the corresponding semi-Galilean algebra is also finite dimensional.

The paper is organized as follows. In the next section we study the Galilean algebra for arbitrary n. In section three we explore how the AdS/CFT correspondence works in this context. The last section is devoted to discussions.

2 General contraction of conformal algebra

In this section we study non-relativistic limit of relativistic conformal algebra in $d+1$ dimensions by making use of a contraction. To proceed we consider the following general scaling

$$t \to t, \qquad y_\alpha \to y_\alpha, \qquad x_i \to \epsilon x_i, \tag{2.1}$$

where $\alpha = 1, \cdots, n$ and $i = n+1, \cdots d$. The contraction is defined by the above scaling in the limit of $\epsilon \to 0$. For $n=0$ this has been studied in [33] where it was shown that the resultant contracted algebra admits an infinite dimensional extension containing one copy of Virasoro algebra. In what follows we would like to extend this consideration for general n.

2.1 Field theory description

We start from a CFT in $d+1$ dimensions. The theory is invariant under the action of generators of conformal algebra given by rotations $J_{\mu\nu}$, translations P_μ, dilation D and

[1] We note, however, that calling this theory a non-relativistic CFT is somehow misleading as it has relativistic properties in those directions which remained unchanged

special conformal transformations K_μ whose representations as a vector field acting on the $d+1$ dimensional Minkowski space are given by

$$J_{\mu\nu} = -(x_\mu \partial_\nu - x_\nu \partial_\mu), \quad P_\mu = \partial_\mu, \quad D = -(x \cdot \partial), \quad K_\mu = -(2x_\mu (x \cdot \partial) - (x \cdot x)\partial_\mu) \tag{2.2}$$

with $\mu = 0, \cdots, d$. The aim is to contract the conformal algebra generated by the above generators. To do this we will consider the general scaling (2.1) in the limit of $\epsilon \to 0$. To be specific we will consider the case of $n = 1$. It is, of course, straightforward to generalize it for arbitrary n.

For $n = 0$ under the above rescaling and in the of $\epsilon \to 0$ the $d+1$ dimensional conformal algebra reduces to Galilean conformal algebra studied in [33]. For $n = 1$ the generators of the corresponding algebra as a vector field acting on a $d+1$ dimensional Minkowski space (for $d \geq 2$) are given by

$$\begin{aligned}
&J_{ij} = -(x_i\partial_j - x_j\partial_i), \quad J_i = t\partial_i, \quad \tilde{J}_i = -y\partial_i, \quad P_i = \partial_i, \quad K_i = (y^2 - t^2)\partial_i, \\
&D = -(x_i\partial_i + y\partial_y + t\partial_t), \quad \tilde{D} = t\partial_y + y\partial_t, \quad P = \partial_t, \quad \tilde{P} = \partial_y, \\
&K = (t^2 + y^2)\partial_t + 2ty\partial_y + 2tx_i\partial_i, \quad \tilde{K} = -(t^2 + y^2)\partial_y - 2ty\partial_t - 2yx_i\partial_i.
\end{aligned} \tag{2.3}$$

To have an insight how the semi-Galilean conformal algebra for $n = 1$ could be, it is useful to define new coordinates $u = t + y$, $v = t - y$ by which the above generators may be recast to the following form[2]

$$\begin{aligned}
&H = 2\partial_u, \quad E = 2(u\partial_u + \frac{1}{2}x_i\partial_i), \quad C = 2(u^2\partial_u + ux_i\partial_i), \\
&\bar{H} = 2\partial_v, \quad \bar{E} = -2(v\partial_v + \frac{1}{2}x_i\partial_i), \quad \bar{C} = 2(v^2\partial_v + vx_i\partial_i), \\
&J_{ij} = -(x_i\partial_j - x_j\partial_i), \quad P_i = \partial_i, \quad B_i = -u\partial_i, \quad \bar{B}_i = v\partial_i, \quad K_i = -uv\partial_i.
\end{aligned} \tag{2.4}$$

It is easy to see that (H, E, C) and $(\bar{H}, \bar{E}, \bar{C})$ generate two copies of $SL(2, R)$ algebra. In fact to write the explicit form of the commutation relations of the algebra it is useful to define $L_{\pm1,0}, \bar{L}_{\pm1,0}$ and $M_{i\,rs}$ $r, s = 0, 1$ as follows

$$\begin{aligned}
&\{L_{-1} = \frac{1}{2}H, \;\; L_0 = \frac{1}{2}E, \;\; L_1 = \frac{1}{2}C\}, \quad \{\bar{L}_{-1} = \frac{1}{2}\bar{H}, \;\; \bar{L}_0 = -\frac{1}{2}\bar{E}, \;\; \bar{L}_1 = \frac{1}{2}\bar{C}\}, \\
&\{M_{i\,00} = -P_i, \;\; M_{i\,01} = B_i, \;\; M_{i\,10} = -\bar{B}_i, \;\; M_{i\,11} = K_i\}.
\end{aligned} \tag{2.5}$$

Using this notation the non-zero commutation relations of the algebra are

$$\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m}, \qquad [\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m}, \\
&[L_n, M_{i\,rs}] = \left(\frac{n+1}{2} - r\right) M_{i\,(n+r)s}, \qquad [\bar{L}_n, M_{i\,rs}] = \left(\frac{n+1}{2} - s\right) M_{i\,s(n+s)},
\end{aligned}$$

[2]For example $H = \tilde{P} + P$, $C = K - \tilde{K}$ and $E = \tilde{D} - D$.

$$[M_{l\,rs}, J_{ij}] = (\delta_{jl} M_{i\,rs} - \delta_{il} M_{j\,rs}), \qquad [J_{ij}, J_{i'j'}] = so(d-1), \tag{2.6}$$

which make the two $SL(2,R)$ subalgebras manifest. Here $n, m = \pm 1, 0$ and $r, s = 0, 1$. Actually it is useful to re-express the generators of (2.5) in the following instructive closed form

$$L_n = u^{n+1}\partial_u + \frac{n+1}{2} u^n x_i \partial_i, \quad \bar{L}_n = v^{n+1}\partial_v + \frac{n+1}{2} v^n x_i \partial_i, \quad M_{i\,rs} = -u^r v^s \partial_i. \tag{2.7}$$

From these expressions it is natural to define the above vector fields for arbitrary integers n and r. Indeed defining

$$J_{ij\,nm} = -u^n v^m (x_i \partial_j - x_j \partial_i), \tag{2.8}$$

one finds an infinite dimensional algebra as follows

$$\begin{aligned}
&[L_n, L_m] = (n-m) L_{n+m}, && [\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m}, \\
&[M_{i\,nm}, M_{j\,n'm'}] = 0, && [L_n, \bar{L}_m] = 0, \\
&[L_n, M_{i\,ml}] = \left(\frac{n+1}{2} - m\right) M_{i\,(n+m)l}, && [\bar{L}_n, M_{i\,ml}] = \left(\frac{n+1}{2} - l\right) M_{i\,m(n+l)} \\
&[L_n, J_{ij\,ml}] = -m J_{ij\,(n+m)l}, && [\bar{L}_n, J_{ij\,ml}] = -l J_{ij\,m(n+l)} \\
&[M_{l\,nm}, J_{ij\,n'm'}] = \left(\delta_{jl} M_{i\,(n+n')(m+m')} - \delta_{il} M_{j\,(n+n')(m+m')}\right).
\end{aligned} \tag{2.9}$$

Moreover the $J_{ij\,nm}$'s generate an $so(d-1)$ affine algebra.

As an conclusion we observed that the (semi) Galilean conformal algebra obtained from the relativistic conformal algebra using the contraction (2.1) admits an infinite dimensional extension for $n = 0, 1$. This may be understood from the fact that in these cases there is at least an $SL(2,R)$ subalgebra which may be extended to a Virasoro algebra. As we will see from gravity point of view this corresponds to the fact that in these cases the gravity dual develops AdS_2 or AdS_3 geometries for $n = 0$ and $n = 1$, respectively.

Actually the above procedure may be generalized for $n \geq 2$ where we will get an algebra containing an $so(2, n+1) \times so(d-n-1)$ subalgebra. We note, however, that in this case the resultant semi-Galilean algebra does not admit an infinite dimensional extension. As we will see in the next section the reason may be understood from the fact that in this case the gravity background develops an AdS_{n+2} geometry which has finite dimensional asymptotic symmetry algebra.

2.2 Gravity description

Following AdS/CFT correspondence we would expect that a $d+1$ dimensional CFT may have a gravity dual defined on a background containing an AdS_{d+2} factor where the CFT lives on the boundary of the AdS space. Therefore one should be able to take the non-relativistic limit from both side of the duality. In particular we would like to carry out the contraction of the previous section on the AdS part of the bulk geometry.

To proceed consider the metric of an AdS_{d+2} space in the Poincaré coordinates

$$ds^2 = \frac{-dt^2 + dy_\alpha^2 + dx_i^2 + dz^2}{z^2}. \tag{2.10}$$

The non-relativistic limit of the previous section can be generalized to the bulk geometry as follows

$$t \to t, \qquad z \to z, \qquad y_\alpha \to y_\alpha, \qquad x_i \to \epsilon x_i. \tag{2.11}$$

In the limit of $\epsilon \to 0$ where only t, z and y_α survive the contraction the resultant geometry develops an AdS_{n+2} space. The rest $d-n$ dimensional space parametrized by x_i are fibered over the AdS_{n+2} base spacetime. Indeed as it was argued in [33] the corresponding gravity dual should be given in terms of the Newton-Cartan like description where AdS_{n+2} plays the special role of the time. In this formalism the metric is non-dynamical and the dynamics are given by torsion free affine connections. More precisely following [33] one may define a contravariant tensor $\gamma = \gamma^{MN}\partial_M \otimes \partial_N$ with $M, N = \{t, z, \alpha, i\}$. It has $n+2$ zero eigenvalues corresponding to $\{t, z, y_\alpha\}$ which parametrize the base AdS_{n+2} space with the metric

$$ds^2 = g_{ab}dx^a dx^b = \frac{-dt^2 + dy_\alpha^2 + dz^2}{z^2}. \tag{2.12}$$

The affine connections Γ^M_{NL} are compatible with both base AdS space as well as the $d-n$ dimensional spatial fiber

$$\nabla_M \gamma^{ML} = 0, \qquad \nabla_M g_{ab} = 0. \tag{2.13}$$

In our case the dynamical connection may be given by $\Gamma^i_{ab} = \partial_i \Phi_{ab}$ [33].

Following the general lore of the AdS/CFT correspondence [49] we would expect that if the above Newtonian gravity provides a gravity dual of the semi-Galilean conformal filed theory one should be able to see the semi-Galilean symmetry algebra as the asymptotic symmetry algebra of the above geometry in the sense of Brown and Henneaux construction [50]. In what follows we will show that this is, indeed, the case. To be specific we will consider the case of $n = 1$ which turns out to be more interesting case. Generalization to other cases is straightforward.

In the Poincaré coordinates the Killing vectors of AdS_{d+2} are given by

$$\begin{aligned} &J_{\mu\nu} = -(x_\mu\partial_\nu - x_\nu\partial_\mu), \qquad\qquad D = -(x^\mu\partial_\mu + z\partial_z), \\ &K_\mu = -(2x_\mu(x^\nu\partial_\nu + z\partial_z) - (x^\nu x_\nu + z^2)\partial_\mu), \qquad P_\mu = \partial_\mu. \end{aligned} \tag{2.14}$$

Using the scaling (2.11) the resultant contracted Killing vectors read

$$\begin{aligned} &P_i = \partial_i, \quad B_i = t\partial_i, \quad \tilde{B}_i = -y\partial_i, \quad K_i = (t^2 - y^2 - z^2)\partial_i, \\ &D = -(t\partial_t + y\partial_y + x_i\partial_i + z\partial_z), \quad \tilde{D} = t\partial_y + y\partial_t, \quad J_{ij} = -(x_i\partial_j - x_j\partial_i), \\ &K = -(t^2 + y^2 + z^2)\partial_t - 2zt\partial_z - 2ty\partial_y - 2tx_i\partial_i, \qquad P = \partial_t, \\ &\tilde{K} = (t^2 + y^2 - z^2)\partial_y + 2ty\partial_t + 2zy\partial_z + 2yx_i\partial_i, \qquad \tilde{P} = \partial_y. \end{aligned} \tag{2.15}$$

Note that to make the comparison more transparent we have used the same labeling for bulk and boundary generators. Following our previous discussions setting $u = t + y, v = t - y$ the above Killing vectors may be recast to the following form

$$
\begin{aligned}
&H = 2\partial_u, \qquad E = 2(u\partial_u + \frac{1}{2}x_i\partial_i + \frac{1}{2}z\partial_z), \qquad C = 2(u^2\partial_u + u(x_i\partial_i + z\partial_z) + z^2\partial_v),\\
&\bar{H} = 2\partial_v, \qquad \bar{E} = -2(v\partial_v + \frac{1}{2}x_i\partial_i + \frac{1}{2}z\partial_z), \qquad \bar{C} = -2(v^2\partial_v + v(x_i\partial_i + z\partial_z) + z^2\partial_u),\\
&J_{ij} = -(x_i\partial_j - x_j\partial_i), \quad P_i = \partial_i, \quad B_i = u\partial_i, \quad \bar{B}_i = v\partial_i, \quad K_i = (uv - z^2)\partial_i. \qquad (2.16)
\end{aligned}
$$

which reduce to those in the previous section in the limit of $z \to 0$ where we approach the boundary of the AdS_3.

Let us define infinite dimensional vector fields in the bulk as follows

$$
\begin{aligned}
L_n &= u^{n+1}\partial_u + \frac{n+1}{2}u^n(x_i\partial_i + z\partial_z) + \frac{n(n+1)}{2}u^{n-1}z^2\partial_v,\\
\bar{L}_n &= v^{n+1}\partial_v + \frac{n+1}{2}v^n(x_i\partial_i + z\partial_z) + \frac{n(n+1)}{2}v^{n-1}z^2\partial_u, \qquad (2.17)
\end{aligned}
$$

which can be properly identified with H, E, C and $\bar{H}, \bar{E}, \bar{C}$ for $n = \pm 1, 0$ which at the boundary where $z \to 0$ reduce to those in (2.7). It is interesting to note that these vector fields asymptotically obey two copies of Virasoro algebra, *i.e.*

$$
[L_n, L_m] = (n - m)L_{n+m}, \quad [\bar{L}_n, \bar{L}_m] = (n - m)\bar{L}_{n+m}, \quad [\bar{L}_n, L_m] = \mathcal{O}(z^4). \qquad (2.18)
$$

The action of the Virasoro generators on the metric of the base manifold, AdS_3, is given

$$
L_n \,:\, ds^2 \to ds^2 + \frac{n(n^2-1)}{2}u^{n-2}du^2, \quad \bar{L}_n \,:\, ds^2 \to ds^2 + \frac{n(n^2-1)}{2}v^{n-2}dv^2. \qquad (2.19)
$$

Therefore the generators of two $SL(2, R)$'s given by $L_{\pm,0}$ and $\bar{L}_{\pm,0}$ are the exact isometries of the base metric, as expected, while for other n's they generate the asymptotic symmetry which preserve the following boundary conditions

$$
\begin{pmatrix}
h_{uu} = \mathcal{O}(1) & h_{uv} = \mathcal{O}(1) & h_{uz} = \mathcal{O}(z)\\
h_{vu} = h_{uv} & h_{vv} = \mathcal{O}(1) & h_{vz} = \mathcal{O}(z)\\
h_{zu} = h_{uz} & h_{zv} = h_{vz} & h_{zz} = \mathcal{O}(1)
\end{pmatrix}. \qquad (2.20)
$$

On the other hand requiring to have asymptotically a closed algebra we will have to extend the other generators as follows

$$
M_{i\,nm} = -(u^n v^m - nmu^{n-1}v^{m-1}z^2)\partial_i, \quad J_{ij\,nm} = -u^n v^m(x_i\partial_j - x_j\partial_i), \qquad (2.21)
$$

which for $m, n = 0, 1$ can be identified with $P_i, B_i, \bar{B}_i, K_i$ and at the boundary where $z \to 0$ reduce to that in (2.7). It is easy to see that

$$
[L_n, M_{i\,lm}] = \left(\frac{n+1}{2} - l\right) M_{i\,(n+l)m} + \mathcal{O}(z^4), \quad [L_n, J_{ij\,lm}] = -lJ_{ij\,(n+l)m} + \mathcal{O}(z^2),
$$

$$[\bar{L}_n, M_{i\,lm}] = \left(\frac{n+1}{2} - m\right) M_{i\,l(n+m)} + \mathcal{O}(z^4), \quad [\bar{L}_n, J_{ij\,lm}] = -mJ_{ij\,l(n+m)} + \mathcal{O}(z^2),$$
$$[M_{l\,nm}, J_{ij\,n'm'}] = \left(\delta_{jl} M_{i\,(n+n')(m+m')} - \delta_{il} M_{j\,(n+n')(m+m')}\right) + \mathcal{O}(z^2),$$
$$[M_{i\,nm}, M_{j\,n'm'}] = 0. \tag{2.22}$$

As a conclusion we have demonstrated that the asymptotic symmetry algebra of our bulk geometry is the semi-Galilean conformal algebra studied in the previous section.

It is straightforward to generalize the above considerations for $n \geq 2$ where the base space will be AdS_{n+2}. In this case from the base space we find an $so(2, n+1)$ factor while from the fiber one gets an $so(d-n-1)$ subalgebra which is compatible with those studies in the previous section. We note that for the case of $n \geq 2$ the semi-Galilean conformal algebra is finite dimensional due to the fact the the asymptotic symmetry of AdS_{n+2} for $n \geq 2$ is finite dimensional.

3 AdS/CFT description of theory with Galilean conformal symmetry

In this section we would like to study the AdS/CFT correspondence for the Galilean conformal field theory. Following the relativistic CFT one would expect that in the Galilean CFT the asymptotic states cannot be defined and the physical observables would be correlation functions. Therefore the task is to compute N-point functions of operators in the Galilean CFT which is the aim of this section.

In what follows we will mainly consider the case of $n = 0$, though the procedure may be generalized for other n's.

3.1 Field theory description

Consider a Galilean CFT in $d+1$ dimensions. As we have seen the corresponding algebra can be obtained from the relativistic CFT by a contraction. Therefore one may naively expect that the N-point functions of Galilean CFT can also be obtained from those in the relativistic CFT by making use of the same contraction. For example consider two point function of an operator ϕ with scaling dimension Δ in a relativistic CFT in $d+1$ dimensions parametrized by t and x_i

$$\langle \phi(t_1, x_i) \phi(t_2, y_i) \rangle \sim \frac{1}{(-(t_1-t_2)^2 + (x_i - y_i)^2)^{\Delta}}. \tag{3.1}$$

Using the scaling limit (2.1) and in the limit of $\epsilon \to 0$ the two point function of the Galilean CFT reads

$$\langle \phi(t_1, x_i) \phi(t_2, y_i) \rangle \sim \frac{1}{(t_1-t_2)^{2\Delta}}. \tag{3.2}$$

Similarly we can extend the above procedure to N-point function to conclude that in general the N-point function of Galilean CFT depends only on time.

We, note, however that although the above results seem reasonable, the way we reach the conclusion may not be correct in general. The reason is due to the fact that the representations of an algebra under a contraction do not necessarily lead to faithful (bijective) representations [51]. In other words although the N-point functions we obtain by this method satisfy the Ward identity of the Galilean CFT, it is not clear that the general form of the N-point functions can be obtained from this method. Therefore it would be interesting to evaluate the N-point function of Galilean CFT directly. To do this we utilize the Ward identity of the Galilean CFT.

The representation of the generators of the Galilean conformal algebra acting on an operator with dimension Δ is given

$$J_{ij} = -(x_i\, p_j - x_j\, p_i), \qquad P_0 = H = -\partial_t, \qquad P_i = \partial_i \qquad B_i = t\partial_i,$$

$$D = -(x_i\partial_i + t\partial_t) - \Delta, \qquad K = -(2tx_i\partial_i + t^2\partial_t) - 2\Delta t, \qquad K_i = t^2\partial_i. \tag{3.3}$$

Note that although the Galilean conformal algebra admits an infinite dimensional extension, we would expect that the vacuum is only invariant under the global part of the algebra given by the above generators. Therefore the Wrad identity of the Galilean CFT may be written as follows

$$\sum_i \langle 0|\phi(x_1)\ldots Q\phi(x_i)\ldots\phi(x_N)|0\rangle = 0 \tag{3.4}$$

where $|0\rangle$ is a vacuum which is invariant under the global part of the algebra. $Q\phi(x_i)$ is the representation of an operator Q on the field $\phi(x_i)$ with Q stands for one of the generators in (3.3). By making use of the equation (3.4) one can write the Ward identities for N-point function $G_N(x_1, t_1, \ldots, x_N, t_N)$. To write the explicit form of the Wrad identities for N-point functions it is useful to define new variables $t_{i2} = t_i - t_2, x_{i2} = x_i - x_2$ and $\tilde{t}_{12} = t_1 + t_2, \tilde{x}_{12} = x_1 + x_2$ for $i = 1, 3, 4\cdots, N$.

From the Ward identities for space and time translations one finds that G_N depends only on t_{i2} and x_{i2}, *i.e.* $G_N(t_{12}, t_{32}, \cdots; x_{12}, x_{32}, \cdots)$. On the other hand from K_i one finds

$$\begin{aligned}&[t_{12}D_{12} + t_{32}D_{32} + \ldots + t_{N2}D_{N2}]G_N = 0,\\ &[t_{32}(t_{32} - t_{12})D_{32} + \ldots + t_{N2}(t_{N2} - t_{12})D_{N2}]G_N = 0,\end{aligned} \tag{3.5}$$

where $D_{i2} = \frac{\partial}{\partial x_{i2}}$. From the dilatation we get

$$[x_{12}D_{12} + \ldots + x_{N2}D_{N2} + t_{12}\partial_{12} + \ldots + t_{N2}\partial_{N2} + \lambda_1 + \ldots + \lambda_N]G_N = 0, \tag{3.6}$$

and K leads to the following differential equation

$$\begin{aligned}&[(2t_{32}x_{32} - t_{32}x_{12} - t_{12}x_{32})D_{32} + \ldots + (2t_{N2}x_{N2} - t_{N2}x_{12} - t_{12}x_{N2})D_{N2}\\ &+t_{32}(t_{32} - t_{12})\partial_{32} + \ldots + t_{N2}(t_{N2} - t_{12})\partial_{N2} + t_{12}(\lambda_1 - \lambda_2 - \ldots - \lambda_N)\\ &+2\lambda_3 t_{32} + \ldots + 2\lambda_N t_N]G_N = 0.\end{aligned} \tag{3.7}$$

Now the task is to solve these equations to find N-point functions. We note, however, that these equations cannot fix the N-point functions completely for arbitrary N. This is of course the case even for relativistic one where the N-point functions can be found up to unknown functions. Let us give the explicit form of two and there point functions.

Two-point function

From (3.5) it is clear that the two-point function does not depend on x_{12} and from (3.6) we get

$$G_2 := \langle\phi_1(t_1, x_{i1})\phi_2(t_2, x_{i2})\rangle = C\ \ t_{12}^{-\Delta}, \qquad\qquad \Delta = \Delta_1 + \Delta_2 \tag{3.8}$$

where C is a constant.

Three-point function

From (3.5) it is clear that the three-point function does not depend on x_{12} and x_{32} and from (3.6) and (3.7) one finds

$$G_3 := \langle\phi_1(t_1, x_{i1})\phi_2(t_2, x_{i2})\phi_3(t_3, x_{i3})\rangle = C(\frac{1}{t_{12}})^{\Delta_1+\Delta_2-\Delta_3}(\frac{1}{t_{32}})^{-\Delta_1+\Delta_2+\Delta_3}(\frac{1}{t_{13}})^{\Delta_1+\Delta_3-\Delta_2} \tag{3.9}$$

N-point function

In principle one could proceed to compute N-point function for arbitrary N, though here we will not do that. The only comment we would like to make is that utilizing the Ward identities one can show that the N-point function depends only on t_{i2}'s.

3.2 Gravity description

In this subsection we would like to see how the N-point functions we have considered in the previous section can be obtained from gravity description. The procedure in the relativistic AdS/CFT correspondence is to evaluate the bulk action on a classical solution with a given boundary condition. Since for Galilean CFT the gravity description is given in terms of the Newtonian gravity the above description may not be directly applied in this case. To explore the procedure we start from a propagating field on an AdS geometry and impose the contraction we have introduced in the previous section.

To proceed, for simplicity, we consider a massive scalar field on the AdS_{d+2} background whose equation of motion is given by

$$\frac{1}{\sqrt{G}}\partial_M\left(\sqrt{G}G^{MN}\partial_N\phi(t,z,x_i)\right) - m^2\phi(t,z,x_i) = 0, \tag{3.10}$$

where G_{MN} is the metric of the AdS geometry. To be specific we consider the AdS geometry in the Poincaré coordinates parametrized by t, z, x_i. Under the scaling (2.11) one has

$$G_{MM} \to G_{MN}, \qquad \partial_t \to \partial_t, \qquad \partial_z \to \partial_z, \qquad \partial_i \to \epsilon^{-1}\partial_i. \tag{3.11}$$

so that

$$\left[\frac{1}{\sqrt{G}}\partial_a\left(\sqrt{G}g^{ab}\partial_b\phi(t,z,x_i)\right) - m^2\phi(t,z,x_i)\right] + \frac{z^2}{\epsilon^2}\partial_i^2\phi(t,z,x_i) = 0. \tag{3.12}$$

Here g_{ab} is the metric of AdS_2 base geometry. In order to have a well behaved equation in the limit of $\epsilon \to 0$ one should impose

$$\frac{1}{\sqrt{G}}\partial_a\left(\sqrt{G}g^{ab}\partial_b\phi(t,z,x_i)\right) - m^2\phi(t,z,x_i) = 0, \qquad \partial_i^2\phi(t,z,x_i) = 0. \tag{3.13}$$

The first equation may be obtained from a two dimensional action given by

$$I = \int dtdz\, \sqrt{G}\, \frac{1}{2}\left(g^{ab}\partial_a\phi\partial_b\phi + m^2\phi^2\right), \tag{3.14}$$

while the second equation may be treated as a constraint. Therefore the most general solution of the equation of motion of the above action is

$$\phi(t,z) = z^{\frac{d+1}{2}}\, e^{-i\omega t}(AI_\alpha(\omega z) + BK_\alpha(\omega z)), \tag{3.15}$$

where $\alpha = \sqrt{\frac{(d+1)^2}{4} + m^2}$. Since in the present case the constraint decouples from the equation of motion, it leads to an overall factor which could depends on x_i[3]. It is then straightforward to follow the general role of the AdS/CFT correspondence to find the bulk solution by given a boundary value as follows

$$\phi(t,z) = c\delta^{\Delta-d-1}\int dt'\phi_\delta(t')\left(\frac{z}{z^2+|t-t'|^2}\right)^\Delta, \tag{3.16}$$

where $\Delta = \frac{d}{2} + \alpha$ and ϕ_δ denotes the Dirichlet boundary value at $z = \delta$. This can be used to read the two point function as follows

$$\langle\mathcal{O}(t_1)\mathcal{O}(t_2)\rangle \sim \frac{1}{(t_1-t_2)^{2\Delta}}. \tag{3.17}$$

in agreement with (3.8) for $\Delta_1 = \Delta_2 = \Delta$.

To find N-point function we should add an interaction term $\lambda_N\phi^N$ to the action. Then following the standard AdS/CFT procedure one arrives at (see for example [52])

$$I_N(t_1,...,t_N) \sim \int dtdz\, \frac{z^{-(d+2)+N\Delta}}{[(z^2+(t-t_1)^2)\dots(z^2+(t-t_N)^2)]^\Delta}. \tag{3.18}$$

In particular for $N = 3$ we get

$$\langle\mathcal{O}(t_1)\mathcal{O}(t_2)\mathcal{O}(t_3)\rangle \sim -\frac{\lambda_3\Gamma\left(\frac{1}{2}\Delta+\alpha\right)}{2\pi^{d+1}}\left[\frac{\Gamma\left(\frac{1}{2}\Delta\right)}{\Gamma(\alpha)}\right]^3\frac{1}{(t_{12}t_{31}t_{23})^\Delta}. \tag{3.19}$$

in agreement with (3.9) for $\Delta_1 = \Delta_2 = \Delta_3 = \Delta$.

[3]We note, however, that the over all factor could parametrically be divergent due to the integration over boundary term. This might be observed by redefinition of the boundary operators by making use of a regularization. The similar behavior happens in the non-relativistic CFT studied in [2, 32].

As a conclusion we have demonstrated how N-point function can be obtained from gravity description of Galilean CFT where we have seen that the main role plays by the base AdS_2 geometry.

Actually the procedure may be summarized as follow. In order to obtain the physical correlation functions one needs to use the standard AdS/CFT correspondence for AdS_2 part, though the action of the corresponding propagating fields in the AdS_2 geometry gets a contribution from the fiber via the measure of the integral. Otherwise the procedure follows the same as that in the standard AdS/CFT correspondence applied for AdS_2.

The above procedure may be generalized to arbitrary n. The only difference is that the AdS_2 has to be replaced by AdS_{n+2}. In other words in this case the action is given by

$$I = \int dtdzd^n y_\alpha \, z^{d-n-1}\sqrt{g}\, \frac{1}{2}(g^{ab}\partial_a\phi\partial_b\phi + m^2\phi^2) \tag{3.20}$$

where g_{ab} is the metric of AdS_{n+2} given by (2.12). Note that the propagating fields are subject to the constraint $\partial_i^2\phi = 0$ for $i = 1, \cdots, d-n-1$.

It is important to note that the decoupling of the fiber is due to the particular form of the constraint. If we change the constraint (for example by breaking the conformal symmetry via heating up the theory) the situation may be changed.

4 Discussions

In this paper we have considered different contractions of a $d+1$ dimensional relativistic conformal algebra. The contraction is defined by the scaling (2.1) in the limit of $\epsilon \to 0$. In other words if we define the velocity of l th direction as

$$v_\alpha = \frac{y_\alpha}{t}, \qquad v_i = \frac{x_i}{t}, \qquad \alpha = 1, \cdots, n, \;\; i = n+1, \cdots, d, \tag{4.1}$$

in this limit one has $v_i \to 0$. Therefore the contraction may be thought of as taking non-relativistic limit of the relativistic conformal algebra. In particular when $n = 0$ the resultant algebra is the Galilean conformal algebra [33]. For $n \geq 1$ we are taking non-relativistic limit in some directions while the others remain unchanged. So, it may be treated as a semi-Galilean conformal algebra.

In general by a contraction the conformal algebra in $d+1$ dimension, $so(2, d+1)$, reduces to an algebra which contains an $so(2, n+1) \times so(d-n-1)$ subalgebra. For $n = 0$ and $n = 1$ the obtained algebras have $SL(2, R)$ and $SL(2, R) \times SL(2, R)$ subalgebra, respectively. Due to this property the corresponding algebras have infinite dimensional extension where the $SL(2, R)$'s extend to the Virasoro algebra. Having had the Virasoro algebra in the cases of $n = 0, 1$, it would be interesting to see if the (semi) Galilean conformal algebra allows a central extension to its Virasoro subalgebra.

Following the AdS/CFT correspondence one may suspect that the (semi) Galilean CFTs may have dual gravity descriptions. If so, the corresponding gravity dual should contain a factor of AdS_{n+2} to support the symmetry group $SO(2, n+1)$. Moreover to have the symmetry group $SO(d-n-1)$ the gravity dual should also have a factor of $d-n-1$ dimensional flat space, $\mathcal{M}_{d-n-1}$. On the other hand since the semi-Galilean conformal

algebra cannot be factorized as $so(2, n+1) \times so(d-n-1)$ the bulk geometry is not a direct product of these two spaces, though locally it may be thought of as $AdS_{n+2} \times \mathcal{M}_{d-n-1}$. In fact it was argued in [33] that at least for $n = 0$ the geometry is a $d-1$ dimensional spatial space fibered over an AdS_2. From our consideration we expect that in general the bulk geometry is a $d-n-1$ dimensional spatial space fibered over an AdS_{n+2}. Note, that, the corresponding gravity is given in terms of Newton-Cartan like description when the role of time is replaced by an AdS_{n+2}.

Using this picture it is easy to understand why the cases of $n = 0$ and 1 have infinite dimensional extension while the other cases are finite dimensional. In fact the reason is due to the asymptotic symmetry of AdS space; while for AdS_2 and AdS_3 it is infinite dimensional, for the others it is finite dimensional[4].

We have also explored the AdS/CFT correspondence for (semi) Galilean CFTs where we have seen that the essential role is played by the base AdS_{n+2} geometry. In fact the correlation functions of the (semi) Galilean CFTs can be evaluated by making use of propagating fields on AdS_{n+2} with proper boundary conditions and a modified measure due to the contribution of the fiber.

An interesting application of this contraction would be to apply the procedure to $\mathcal{N} = 4$ four dimensional SYM theory whose gravity dual is given by type IIB string theory on $AdS_5 \times S^5$. Taking the limit from both sides of the duality one may single out a subset of $\mathcal{N} = 4$ four dimensional SYM theory which has (semi) Galilean conformal symmetry. This might give a new insight about the AdS/CFT correspondence following [53].

In the context of AdS/CFT duality it is known that heating up the dual field theory generically corresponds to adding a black hole in the bulk gravity. Therefore we would expect that applying the above limit one may find a gravity dual to the (semi) Galilean CFT at finite temperature. In this case the bulk gravity background may be interpreted as a $d-n-1$ dimensional spatial space fibered over a base which is given by a black hole in AdS_{n+2} space. It is worth noting that in this case the contraction may by supplemented by a shift in ∂_t. In this case the constraint does not decouple from the equation of motion of propagating modes in the base AdS_{n+2} space. As a result the correlation function will depend on the fiber coordinates too [54].

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[4]It should be compared with Schrödinger algebra which has infinite dimensional extension in any dimension.

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